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THE STRESS FIELD CREATED BY A CIRCULAR SLIDING CONTACT ON TRANSVERSELY ISOTROPIC SPHERES

L. M. Keer and D. B. Mowry
Department of Civil Engineering, Northwestern University, Evanston, IL 60201, U.S.A.

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Abstract—Equations are obtained for the complete stress field due to contact between identical, transversely isotropic, rough spheres loaded normally and tangentially. The stresses are given explicitly for any point in the medium.

INTRODUCTION

This paper considers the state of stress arising when two identical, transversely isotropic, rough spheres are pressed together, first by normal and then by tangential loads. The dimensions of the contact region are considered to be sufficiently small compared with those of the body that the assumptions of Hertz[1], who solved the isotropic problem for the normal loading of smooth spheres, are considered valid. The case of isotropic, rough spheres tangentially loaded was solved by Cattaneo[2] and Mindlin[3]. For their solution the assumption of a contact that remains circular, although slip occurs in an outer annular portion of the region, is found to be valid provided that one allows the relative displacement in the slip region to be unconstrained. The tangential force will cause an axially symmetric tangential stress acting in the direction of the force in both the fixed and slip regions (see Fig. 1). It is the purpose here to extend their problem to the case of transversely isotropic materials and in addition to calculate the stress field within the body.

The stress field created by a circular sliding contact on an isotropic half space has been solved by Hamilton and Goodman[4]. Motivated by considerations of mechanical failure, they examined constant lines of von Mises' yield criteria in the half space and on the surface. Chen[5] and Dahan and Zarka[6] have recently solved for the stress field in a perfectly smooth, transversely isotropic half space in contact with an elastic spherical indenter under normal

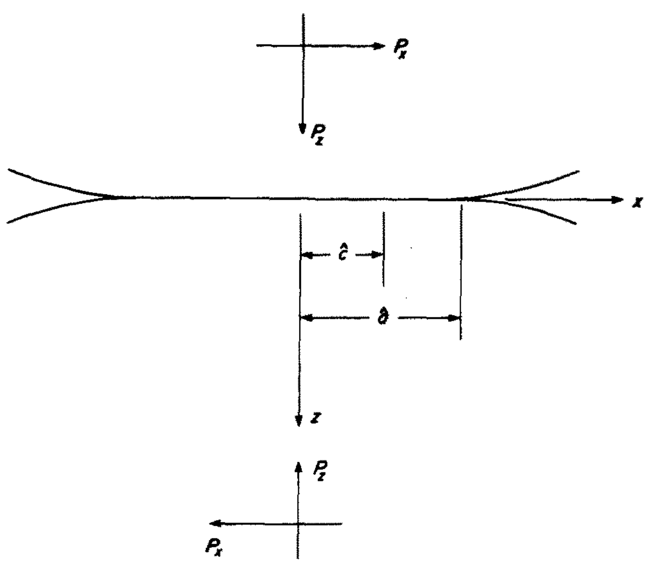

Fig. 1. Geometry and coordinate system.

loading. In both papers results are plotted for several transversely isotropic metals to show the effect of the anisotropy for the indentation of an elastic half space.

By noting that the solution of Dahan and Zarka can be specialized to represent the stress field between two identical, transversely isotropic, normally loaded rough spheres (of sufficiently large radius that the Hertz assumptions are valid), the present analysis will present the expressions for stresses for identical, transversely isotropic, rough spheres having both normal and tangential loading. The case for shear loading has been solved by Chen[5], where the contact loading is a shear stress proportional to the normal stress. The present analysis considers three regions: a region that is bonded, a region in which slip occurs (shear stress proportional to the normal stress), and a region free of stress. The solution, although obtained differently, will be completely analogous to that of Mindlin's.

The approach is to apply first a normal load, P_z, to the two spheres, thereby obtaining the stress field from [6]. Then the solution for an applied tangential load, P_x, is solved and superposed with the stresses due to P_z. Since the normally loaded case is already solved, the method of solution for the tangentially loaded case will be discussed and results for the two superposed cases will be given in the next section. The geometry and coordinate system for the tangentially loaded case are given in Fig. 1.

TANGENTIAL LOADING

The tangential loading of a transversely isotropic half space is considered. The notation used is that given by Green and Zerna[7, pp. 177–180]. For transverse isotropy the relations between stress and displacement are given as

$$\begin{aligned}
\sigma_{xx} &= c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y} + c_{13}\frac{\partial u_z}{\partial z}\\
\sigma_{yy} &= c_{12}\frac{\partial u_x}{\partial x} + c_{11}\frac{\partial u_y}{\partial y} + c_{13}\frac{\partial u_z}{\partial z}\\
\sigma_{zz} &= c_{13}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + c_{33}\frac{\partial u_z}{\partial z}\\
\sigma_{yz} &= c_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right),\ \sigma_{zx} = c_{44}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right),\ \sigma_{xy} = \frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right).
\end{aligned} \tag{1}$$

Substitution of eqns (1) into the equilibrium equations leads to the following equations governed by the potentials ϕ:

$$\begin{aligned}
&(c_{13} + c_{44} + kc_{44})\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + kc_{33}\frac{\partial^2\phi}{\partial z^2} = 0,\\
&c_{11}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + [c_{44} + k(c_{13} + c_{44})]\frac{\partial^2\phi}{\partial z^2} = 0,\\
&-\frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial^2\phi_3}{\partial x^2} + \frac{\partial^2\phi_3}{\partial y^2}\right) - c_{44}\frac{\partial^2\phi_3}{\partial z^2} = 0,
\end{aligned} \tag{2}$$

where

$$u_x = \frac{\partial\phi}{\partial x} + \frac{\partial\phi_3}{\partial y},\ u_y = \frac{\partial\phi}{\partial y} - \frac{\partial\phi_3}{\partial x},\ u_z = k\frac{\partial\phi}{\partial z}. \tag{3}$$

Let

$$u_x = \frac{\partial}{\partial x}(\phi_1 + \phi_2) + \frac{\partial\phi_3}{\partial y},\ u_y = \frac{\partial}{\partial y}(\phi_1 + \phi_2) - \frac{\partial\phi_3}{\partial x},\ u_z = \frac{\partial}{\partial z}(k_1\phi_1 + k_2\phi_2) \tag{4}$$

then it is easily verified that eqns (2) are satisfied by the potentials written in the following form

$$\phi_l = \int_0^\infty A_l(\xi)\, e^{-\xi z/\sqrt{(\nu_l)}} J_1(\xi r)\, d\xi \cos\theta,\ l = 1,2,$$

$$\phi_3 = \int_0^\infty A_3(\xi)\, e^{-\xi z/\sqrt{(\nu_3)}} J_1(\xi r)\, d\xi \sin\theta \tag{5}$$

where

$$\nu_1 = -[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}] - \{[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}]^2 - 4c_{11}c_{33}c_{44}^2\}^{1/2}/2c_{11}c_{44},$$
$$\nu_2 = -[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}] + \{[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}]^2 - 4c_{11}c_{33}c_{44}^2\}^{1/2}/2c_{11}c_{44},$$
$$\nu_3 = 2c_{44}/(c_{11}-c_{12}). \tag{6}$$

Also denote

$$k_1 = (c_{11}\nu_1 - c_{44})/(c_{13}+c_{44}),\ k_2 = (c_{11}\nu_2 - c_{44})/(c_{13}+c_{44}). \tag{7}$$

Here, in anticipation of the tangentially loaded problem, symmetry with respect to the x-axis has been assumed.

When normal loading is applied, there exists a circular region of contact, $0 < r < \hat{a}$, between the spheres. Subsequently, upon application of the tangential loading, P_x, the contact region is assumed to remain circular (see, e.g. [2, 3]). The contact region is divided into two zones. There is an inner zone, $0 < r < \hat{c}$, in which no relative slip occurs, and a zone, $\hat{c} < r < \hat{a}$, in which slip occurs (Fig. 1). It is clear that the tangential loading can be separated from the normally loaded case.

The boundary conditions for the tangentially loaded case on the surface, $z = 0$, may be written as

$$\left.\begin{aligned} u_x &= \alpha \\ u_y &= 0 \end{aligned}\right\} \quad 0 < r < \hat{c},$$

$$\left.\begin{aligned} \sigma_{zx} &= f'\rho_0(\hat{a}^2 - r^2)^{1/2}/\hat{a} \\ \sigma_{zy} &= 0 \end{aligned}\right\} \quad \hat{c} < r < \hat{a}, \tag{8}$$

$$\sigma_{zx} = \sigma_{zy} = 0 \qquad \hat{a} < r < \infty,$$

$$\sigma_{zz} = 0 \qquad 0 < r < \infty,$$

where α is a constant, f' is the coefficient of friction of the surface of the spheres, and ρ_0 is the pressure at the center of the contact zone due to the normal loads, P_z.

The normal stress distribution, given as

$$\rho(r) = \rho_0(\hat{a}^2 - r^2)^{1/2}/\hat{a} \tag{9}$$

where

$$\rho_0 = 3P_z/2\pi\hat{a}^2, \tag{10}$$

is the well known Hertz expression for the pressure distribution between two spheres in contact under normal loading. If the Coulomb friction law is assumed to be applicable, then it is clear that the boundary condition for slip given in eqns (8) is appropriate. It is noted that the displacements in the slip region may not be in the same direction as the tangential shears.

With the boundary conditions, eqns (8), the stresses and displacements in terms of potential functions as given in [7] are transformed into a polar-cylindrical form. The boundary value problem involves simultaneous pairs of dual integral equations which can be solved by the method of Westmann[8, 9]. The problem is analogous to that solved by Goodman and Keer[10], where the relation between the constant, α, and $f'\rho_0$ is obtained by the requirement that a shear stress singularity be required to vanish. The details of the reduction to coupled pairs of dual integral equations and the calculation of these quantities will not be included in this paper, since, except for the change of elastic constants from isotropic to transversely isotropic, the form is the same and the solution technique is an obvious extension of previous work.

The relation between $f'\rho_0$ and $2\alpha = \Delta$ is found to be

$$\Delta = -f'\rho_0 Ge\pi(\hat{a}^2 - \hat{c}^2) \tag{11}$$

where

$$e = \frac{1}{2}(1 + ac/b),\ G = \surd(\nu_3)/2c_{44}\hat{a},$$

$$K = \left(\frac{k_1 c_{33}}{\nu_1} - c_{13}\right) \Big/ \left(c_{13} - \frac{k_2 c_{33}}{\nu_2}\right), \tag{12}$$

$$a = 1 + K,\ b = c_{44}[(1+k_1)\nu_1^{-1/2} + K(1+k_2)\nu_2^{-1/2}],\ c = c_{44}/\nu_3^{1/2}.$$

An expression for the radius of contact, $\hat{a}$, as in [5] can be determined as

$$2\hat{a} = [3P_z R(\delta_1 - \delta_2)]^{1/3} \tag{13}$$

where R is the radius of the sphere and δ_i, $i = 1,2$, are defined in Appendix 1.

Having solved the boundary value problem, the stress expressions within the spheres due to tangential loading are obtained and superposed with those of Dahan and Zarka[6] for the normally loaded case. The results and notation of Dahan and Zarka are used extensively, especially the closed form solution of certain key integrals. Notation used in the stress expressions is given in Appendices 1 and 2, where Appendix 1 gives the relations among the elastic constants and Appendix 2 gives the significant integral identities.

SUPERPOSITION SOLUTION WITHIN SPHERE

The superposition of the stresses due to normal and tangential loading yields stresses in the spheres as follows:

$$\begin{aligned}\sigma_{rr}/\rho_0 = 4f'G\cos\theta\{&L_j(c_{13}k_j\nu_j^{-1} - c_{11})(\hat{a}D_{j2}(\hat{a}) - \hat{c}D_{j2}(\hat{c}))\\ &+ L_i(c_{11} - c_{12})[(\hat{a}D_{i4}(\hat{a}) - \hat{c}D_{i4}(\hat{c}))r^{-2} - (\hat{a}D'_{i4}(\hat{a}) - \hat{c}D'_{i4}(\hat{c}))r^{-1}]\}\\ &-\{[s_1C_{12}(\hat{a}) - s_2C_{22}(\hat{a})](d')^{-1/2} - \nu r^{-1}[s_1\rho_2D_{13}(\hat{a}) - s_2\rho_1D_{23}(\hat{a})]\}/(s_1 - s_2)\end{aligned} \tag{14}$$

where $i = 1,2,3$ summed, $j = 1,2$ summed, and primed notation (with the exception of f', a') denotes differentiation with respect to r.

$$\begin{aligned}\sigma_{\theta\theta}/\rho_0 = 4f'G\cos\theta\{&L_j(c_{13}k_j\nu_j^{-1} - c_{12})(\hat{a}D_{j2}(\hat{a}) - \hat{c}D_{j2}(\hat{c}))\\ &+ L_i(c_{12} - c_{11})[(\hat{a}D_{i4}(\hat{a}) - \hat{c}D_{i4}(\hat{c}))r^{-2} - (\hat{a}D'_{i4}(\hat{a}) - \hat{c}D'_{i4}(\hat{c}))r^{-1}]\}\\ &+\{\surd(d')[s_1q_2C_{12}(\hat{a}) - s_2q_1C_{22}(\hat{a})](a'c' - d')^{-1} - \nu r^{-1}[s_1\rho_2D_{13}(\hat{a}) - s_2\rho_1D_{23}(\hat{a})]\}/(s_1 - s_2)\end{aligned} \tag{15}$$

$$\begin{aligned}\sigma_{r\theta}/\rho_0 = 4f'G(c_{11} - c_{12})\sin\theta\Big\{&\frac{1}{2}L_3(\hat{a}D_{32}(\hat{a}) - \hat{c}D_{32}(\hat{c}))\\ &+ L_i[(\hat{a}D_{i4}(\hat{a}) - \hat{c}D_{i4}(\hat{c}))r^{-2} - (\hat{a}D'_{i4}(\hat{a}) - \hat{c}D'_{i4}(\hat{c}))r^{-1}]\Big\}\end{aligned} \tag{16}$$

$$\sigma_{z\theta}/\rho_0 = 4f'Gc_{44}\sin\theta\{L_j(1+k_j)\nu_j^{-1/2}(\hat{a}D_{j3}(\hat{a}) - \hat{c}D_{j3}(\hat{c}))r^{-1} - L_3\nu_3^{-1/2}(\hat{a}D'_{33}(\hat{a}) - \hat{c}D'_{33}(\hat{c}))\} \tag{17}$$

$$\begin{aligned}\sigma_{rz}/\rho_0 = 4f'Gc_{44}\cos\theta\{&L_j(-1 - k_j)\nu_j^{-1/2}(\hat{a}D'_{j3}(\hat{a}) - \hat{c}D'_{j3}(\hat{c}))\\ &+ L_3\nu_3^{-1/2}(\hat{a}D_{33}(\hat{a}) - \hat{c}D_{33}(\hat{c}))r^{-1}\} + [D_{12}(\hat{a}) - D_{22}(\hat{a})]/[(s_1 - s_2)\surd(d')]\end{aligned} \tag{18}$$

$$\sigma_{zz}/\rho_0 = 4f'G\cos\theta\{L_j(c_{33}k_j\nu_j^{-1} - c_{13})(\hat{a}D_{j2}(\hat{a}) - \hat{c}D_{j2}(\hat{c}))\} + [s_2C_{12}(\hat{a}) - s_1C_{22}(\hat{a})]/(s_1 - s_2). \tag{19}$$

For the surface stresses, the exponential term in the potential functions is unity and the integrals can be easily evaluated as appropriate Hankel transforms. Performing the necessary re-evaluation, the surface stresses can be written as

$0 < r < \hat{c}$:

$$\sigma_{rr}/\rho_0 = \left\{-\frac{1}{\sqrt{d'}}\left(1-\frac{r^2}{\hat{a}^2}\right)^{1/2} + \frac{\mu\hat{a}^2}{3r^2}\left[1-\left(1-\frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\} \tag{20}$$

$$\sigma_{\theta\theta}/\rho_0 = \left\{\left(\mu-\frac{1}{\sqrt{d'}}\right)\left(1-\frac{r^2}{\hat{a}^2}\right)^{1/2} - \frac{\mu\hat{a}^2}{3r^2}\left[1-\left(1-\frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\} \tag{21}$$

$$\sigma_{r\theta} = 0 \tag{22}$$

$$\sigma_{rz}/\rho_0 = 2cf'G\cos\theta\{(\hat{a}^2 - r^2)^{1/2} - (\hat{c}^2 - r^2)^{1/2}\} \tag{23}$$

$$\sigma_{z\theta}/\rho_0 = -2cf'G\sin\theta\{(\hat{a}^2 - r^2)^{1/2} - (\hat{c}^2 - r^2)^{1/2}\} \tag{24}$$

$$\sigma_{zz}/\rho_0 = -(1-(r/\hat{a})^2)^{1/2}. \tag{25}$$

$\hat{c} < r < \hat{a}$:

$$\begin{aligned}\sigma_{rr}/\rho_0 = 2f'G\cos\theta&\left\{L_j(c_{13}k_j\nu_j^{-1} - c_{11})\left[\frac{1}{2}\pi r - r\sin^{-1}(\hat{c}/r) + (\hat{c}/r)(r^2-\hat{c}^2)^{1/2}\right]\right.\\ &\left.+ L_i\left(\frac{c_{11}-c_{12}}{r}\right)\left[\frac{\pi r^2}{8} - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{\hat{c}}{r^2}\right)(r^2-\hat{c}^2)^{1/2}\left(\frac{r^2}{4}-\frac{\hat{c}^2}{2}\right)\right]\right\}\\ &+\left\{-\frac{1}{\sqrt{d'}}\left(1-\frac{r^2}{\hat{a}^2}\right)^{1/2} + \frac{\mu\hat{a}^2}{3r^2}\left[1-\left(1-\frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\}\end{aligned} \tag{26}$$

$$\begin{aligned}\sigma_{\theta\theta}/\rho_0 = 2f'G\cos\theta&\left\{L_j\left(\frac{c_{13}k_j}{\nu_j} - c_{12}\right)\left[\frac{\pi r}{2} - r\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \frac{\hat{c}}{r}(r^2-\hat{c}^2)^{1/2}\right]\right.\\ &\left.+ L_i\left(\frac{c_{12}-c_{11}}{r}\right)\left[\frac{\pi r^2}{8} - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{\hat{c}}{r^2}\right)(r^2-\hat{c}^2)^{1/2}\left(\frac{r^2}{4}-\frac{\hat{c}^2}{2}\right)\right]\right\}\\ &+\left\{\left(\mu-\frac{1}{\sqrt{d'}}\right)\left(1-\frac{r^2}{\hat{a}^2}\right)^{1/2} - \frac{\mu\hat{a}^2}{3r^2}\left[1-\left(1-\frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\}\end{aligned} \tag{27}$$

$$\begin{aligned}\sigma_{r\theta}/\rho_0 = 2f'G\sin\theta&\left\{-L_3\left(\frac{c_{11}-c_{12}}{2}\right)\left[\frac{\pi r}{2} - r\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \frac{\hat{c}}{r}(r^2-\hat{c}^2)^{1/2}\right]\right.\\ &\left.+ L_i\left(\frac{c_{11}-c_{12}}{r}\right)\left[\frac{\pi r^2}{8} - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{\hat{c}}{r^2}\right)(r^2-\hat{c}^2)^{1/2}\left(\frac{r^2}{4}-\frac{\hat{c}^2}{2}\right)\right]\right\}\end{aligned} \tag{28}$$

$$\sigma_{rz}/\rho_0 = 2cf'G(\hat{a}^2 - r^2)^{1/2}\cos\theta \tag{29}$$

$$\sigma_{z\theta}/\rho_0 = -2cf'G(\hat{a}^2 - r^2)^{1/2}\sin\theta \tag{30}$$

$$\sigma_{zz}/\rho_0 = -[1-(r/\hat{a})^2]^{1/2}. \tag{31}$$

$\hat{a} < r < \infty$:

$$\begin{aligned}\sigma_{rr}/\rho_0 = 2f'G\cos\theta&\left\{L_j\left(\frac{c_{13}k_j}{\nu_j} - c_{11}\right)\left[r\sin^{-1}\left(\frac{\hat{a}}{r}\right) - r\sin^{-1}\left(\frac{\hat{c}}{r}\right) - \frac{\hat{a}}{4r}(r^2-\hat{a}^2)^{1/2}\right.\right.\\ &\left.+(r^2-\hat{c}^2)^{-1/2}\left(cr - \frac{5c^3}{4r}\right)\right] + L_i\left(\frac{c_{11}-c_{12}}{r}\right)\left[\frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right.\\ &\left.\left.-\frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{\hat{c}}{4}-\frac{\hat{c}^3}{2r^2}\right)(r^2-\hat{c}^2)^{1/2} + \left(\frac{\hat{a}^3}{2r^2}-\frac{\hat{a}}{4}\right)(r^2-\hat{a}^2)^{1/2}\right]\right\} + \frac{\mu}{3}\cdot\frac{\hat{a}^2}{r^2}\end{aligned} \tag{32}$$

$$\sigma_{\theta\theta}/p_0 = 2f'G\cos\theta\left\{L_j\left(\frac{c_{13}k_j}{\nu_j} - c_{12}\right)\left[r\sin^{-1}\left(\frac{\hat{a}}{r}\right) - r\sin^{-1}\left(\frac{\hat{c}}{r}\right) - \frac{\hat{a}}{4r}(r^2-\hat{a}^2)^{1/2}\right.\right.$$
$$+ (r^2-\hat{c}^2)^{-1/2}\left.\left(\hat{c}r - \frac{5\hat{c}^3}{4r}\right)\right] + L_i\left(\frac{c_{12}-c_{11}}{r}\right)\left[\frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right.$$
$$\left.\left. - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{\hat{c}}{4} - \frac{\hat{c}^3}{2r^2}\right)(r^2-\hat{c}^2)^{1/2} + \left(\frac{\hat{a}^3}{2r^2} - \frac{\hat{a}}{4}\right)(r^2-\hat{a}^2)^{1/2}\right]\right\} - \frac{\mu}{3}\cdot\frac{\hat{a}^2}{r^2} \tag{33}$$

$$\sigma_{r\theta}/p_0 = 2f'G\sin\theta\left\{-L_3\left(\frac{c_{11}-c_{12}}{2}\right)\left[r\sin^{-1}\left(\frac{\hat{a}}{r}\right) - r\sin^{-1}\left(\frac{\hat{c}}{r}\right) - \frac{\hat{a}}{4r}(r^2-\hat{a}^2)^{1/2}\right.\right.$$
$$+ (r^2-\hat{c}^2)^{-1/2}\left.\left(\hat{c}r - \frac{5\hat{c}^3}{4r}\right)\right] + L_i\left(\frac{c_{11}-c_{12}}{r}\right)\left[\frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right) - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right.$$
$$\left.\left. + \left(\frac{\hat{c}}{4} - \frac{\hat{c}^3}{2r^2}\right)(r^2-\hat{c}^2)^{1/2} + \left(\frac{\hat{a}^3}{2r^2} - \frac{\hat{a}}{4}\right)(r^2-\hat{a}^2)^{1/2}\right]\right\} \tag{34}$$

$$\sigma_{rz} = \sigma_{z\theta} = \sigma_{zz} = 0. \tag{35}$$

By equilibrium considerations it can be found that the relationship between the load, P_x, and the horizontal displacement of the sphere, $\Delta/2$, is

$$-P_x = 2c\left[-\frac{\Delta}{e}\hat{c} - p_0 f' G\pi\left(\hat{a}^2\hat{c} - \frac{2\hat{a}^3}{3} - \frac{\hat{c}^3}{3}\right)\right]. \tag{36}$$

The stress fields for the case where the spheres slide relative to one another, as e.g. in [4] when $P_x = f'P_z$, is easily obtained by specializing the above results for $\hat{c}\to 0$.

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APPENDIX 1

Nomenclature

$a_{11}, a_{12}, a_{13}, a_{33}, a_{44}$ = elastic compliances of given transversely isotropic spheres, as used in [5]

$c_{11}, c_{12}, c_{13}, c_{33}, c_{44}$ = elastic moduli of given transversely isotropic spheres, as used in [4]. (See Table 1.)

$a' = a_{13}(a_{11}-a_{12})/(a_{11}a_{33}-a_{13}^2)$

$b' = [a_{13}(a_{13}+a_{44}) - a_{12}a_{33}]/(a_{11}a_{33}-a_{13}^2)$

$c' = [a_{13}(a_{11}-a_{12}) + a_{11}a_{44}]/(a_{11}a_{33}-a_{13}^2)$

$d' = (a_{11}^2 - a_{12}^2)/(a_{11}a_{33}-a_{13}^2)$

$L_1 = -c/2b,\ L_2 = -Kc/2b,\ L_3 = 1/2$

$q_1 = (b' - a's_2^2)\rho_1,\ q_2 = (b' - a's_1^2)\rho_2$

$s_1 = \{[a' + c' + \sqrt{((a'+c')^2 - 4d')}]/2d'\}^{1/2} = \nu_1^{-1/2}$

$s_2 = \{[a' + c' - \sqrt{((a'+c')^2 - 4d')}]/2d'\}^{1/2} = \nu_2^{-1/2}$

$\alpha_i^2 = \frac{1}{2}(r^2 - \hat{m}^2 + s_i^2z^2 + \hat{m}^2\gamma_i^2)$, where $\hat{m} = \hat{a}$ or $\hat{c}$

$$\beta_i = -s_i z\hat{m}/\alpha_i$$
$$\gamma_i^4 = \hat{m}^{-4}(r^2 - \hat{m}^2 + s_i^2 z^2)^2 + 4s_i^2 z^2 \hat{m}^{-2}$$
$$\delta_1 = a_{44}\frac{s_1 s_2}{s_2 - s_1} + (a_{12} - a_{11})\frac{\nu s_1^2 \rho_2}{s_2 - s_1}$$
$$\delta_2 = a_{44}\frac{s_1 s_2}{s_2 - s_1} + (a_{12} - a_{11})\frac{\nu s_2^2 \rho_1}{s_2 - s_1}$$
$$\mu = (b' - 1)(a' + \sqrt{(d')})/(a'c' - d')$$
$$\nu = (b' - 1)\sqrt{(d')}/(a'c' - d')$$
$$\rho_1 = 1 - a's_1^2,\ \rho_2 = 1 - a's_2^2.$$

APPENDIX 2

Integral expressions

$$C_{i2}(\hat{m}) = \int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^2} - \frac{\cos(\hat{m}\xi)}{\xi}\right] J_0(\xi r)\, e^{-\xi z s_i}\, d\xi$$
$$= 1 - \frac{1}{\hat{m}}\left(rS_{i,1}^1(\hat{m}) + s_i z S_{i,1}(\hat{m})\right)$$

$$C_{i3}(\hat{m}) = \int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^3} - \frac{\cos(\hat{m}\xi)}{\xi^2}\right] J_0(\xi r)\, e^{-\xi z s_i}\, d\xi$$
$$= \frac{1}{2\hat{m}}\left(\hat{m}^2 - \frac{r^2}{2} + s_i^2 z^2\right) S_{i,1}(\hat{m}) + \frac{3s_i z r}{4\hat{m}} S_{i,1}^1(\hat{m}) + \frac{r}{4} T_{i,1}^1(\hat{m}) - \frac{s_i z}{2}$$

$$D_{i2}(\hat{m}) = \int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^2} - \frac{\cos(\hat{m}\xi)}{\xi}\right] J_1(\xi r)\, e^{-\xi z s_i}\, d\xi$$
$$= \frac{1}{2\hat{m}}(rS_{i,1}(\hat{m}) - s_i z S_{i,1}^1(\hat{m}) - \hat{m} T_{i,1}^1(\hat{m}))$$

$$D_{i3}(\hat{m}) = \int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^3} - \frac{\cos(\hat{m}\xi)}{\xi^2}\right] J_1(\xi r)\, e^{-\xi z s_i}\, d\xi$$
$$= \frac{1}{3\hat{m}}\left(\hat{m}^2 - r^2 + \frac{s_i^2 z^2}{2}\right) S_{i,1}^1(\hat{m}) + \frac{s_i z}{6} T_{i,1}^1(\hat{m}) - \frac{r s_i z}{2\hat{m}} S_{i,1}(\hat{m}) + \frac{r}{3}$$

$$D_{i4}(\hat{m}) = \int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^4} - \frac{\cos(\hat{m}\xi)}{\xi^3}\right] J_1(\xi r)\, e^{-\xi z s_i}\, d\xi$$
$$= \frac{\hat{m}^2}{4}(D_{i2}(\hat{m}) + T_{i,1}^1(\hat{m})) + \frac{r}{4} C_{i3}(\hat{m}) - \frac{s_i z}{4} D_{i3}(\hat{m})$$

$$S_{i,1}(\hat{m}) = \int_0^\infty \frac{\sin(\hat{m}\xi)}{\xi} J_0(\xi r)\, e^{-\xi z s_i}\, d\xi = \arctan\left(\frac{\hat{m} - \beta_i}{\alpha_i + s_i z}\right)$$

$$S_{i,1}^1(\hat{m}) = \int_0^\infty \frac{\sin(\hat{m}\xi)}{\xi} J_1(\xi r)\, e^{-\xi z s_i}\, d\xi = \frac{\beta_i + \hat{m}}{r}$$

$$T_{i,1}^1(\hat{m}) = \int_0^\infty \frac{\cos(\hat{m}\xi)}{\xi} J_1(\xi r)\, e^{-\xi z s_i}\, d\xi = \frac{\alpha_i - s_i z}{r} \quad .$$